JOURNAL OF COMBINATORIAL THEORY, Series A 29, 267–279 (1980)

On Unimodal Subsequences

F. R. K. CHUNG

Bell Laboratories, Murray Hill, New Jersey 07974

Communicated by the Managing Editors

Received October 25, 1979

In this paper we prove that any sequence of n real numbers contains a unimodal subsequence of length at least $\lceil (3n-3/4)^{1/2}-\frac{1}{2}\rceil$ and that this bound is best possible.

I. INTRODUCTION

Let p denote a permutation on the integers $\{1,..., n\}$. A subsequence is defined to be a subset $\{a_1 < a_2 < \cdots < a_t\}$, where $a_i \in \{1, 2,..., n\}$ for $1 \leqslant i \leqslant t$. A subsequence of p, denoted by $\{a_1 < a_2 < \cdots < a_t\}$, is said to be increasing if

$$p(a_1) < p(a_2) < \cdots < p(a_t).$$

A subsequence $\{a_1 < a_2 < \cdots < a_t\}$ is said to be decreasing if

$$p(a_1) > p(a_2) > \cdots > p(a_t).$$

A subsequence is said to be monotone if it is either increasing or decreasing. A well-known result of Erdős and Szekeres [4] states that any permutation on $\{1, 2,..., n\}$ contains a monotone subsequence of length $n^{1/2}$.

A subsequence $\{a_1 < a_2 < \cdots a_t\}$ is said to be strongly unimodal if, for some k, we have

$$p(a_1) < p(a_2) < \cdots < p(a_k) > p(a_{k+1}) > \cdots > p(a_t).$$

It can be shown that any ermutation on $\{1,..., n\}$ contains a strong unimodal sequence of length $\lceil (2n+1/4)^{1/2}-\frac{1}{2}\rceil$ by a simple proof which has been found by J. M. Steele, and V. Chvátal (among others, although it is unpublished) and is involved in the proof of the main result in this paper. A subsequence $\{a_1 < a_2 < \cdots < a_t\}$ is said to be unimodal if, for some k, we have either

$$p(a_1) < p(a_2) < \cdots < p(a_k) > p(a_{k+1}) > \cdots > p(a_t)$$

0097-3165/80/060267–13$02.00/0

or

$$p(a_1) > p(a_2) > \cdots > p(a_k) < p(a_{k+1}) > \cdots > p(a_t).$$

In this paper we settle a conjecture of Steele [9] by showing that any permutation on $\{1,..., n\}$ contains a unimodal subsequence of length $\lceil (3n-3/4)^{1/2} - \frac{1}{2} \rceil$ and that this is best possible. Suppose p is a mapping from $\{1,..., n\}$ to real numbers, i.e., p is a sequence of real numbers (not necessarily distinct) of length n. It follows immediately from our main result in this paper that there is a unimodal subsequence of length at least $\lceil (3n-3/4)^{1/2} - \frac{1}{2} \rceil$.

II. Preliminaries

First, we will make a few useful definitions. Let p be a fixed permutation on $\{1,..., n\}$. For a number $m \in \{1,..., n\}$, we define $x(m)$ to be the maximum length of an increasing subsequence of p ending at m, i.e.,

$$x(m) = \max\{t: a_1 < a_2 < \cdots < a_t = m \text{ and } p(a_1) < p(a_2) < \cdots < p(a_t)\}.$$

We define $y(m)$ to be the maximum length of a decreasing subsequence of p starting at m, i.e.,

$$y(m) = \max\{t: m = a_1 < a_2 < \cdots < a_t \text{ and } p(a_1) > p(a_2) > \cdots > p(a_t)\}.$$

Similarly, we define $z(m)$ to be the maximum length of an increasing subsequence of p starting at m and $w(m)$ to be the maximum length of a decreasing subsequence of p ending at m.

Let $\rho(p)$ denote the maximum length of a unimodal subsequence in p. It is rather straightforward to verify the following fact.

FACT 1. *Let N denote*

$$N = \max_{1 \leqslant i \leqslant n} \{x(i) + y(i) - 1,\ z(i) + w(i) - 1\}.$$

Then we have

$$N = \rho(p).$$

FACT 2. *Suppose $x(m) = x(m')$ and $m < m'$. Then we have $p(m) > p(m')$ and $y(m) \neq y(m')$, $w(m) \neq w(m')$.*

Proof. Suppose $p(m) < p(m')$. We then have $x(m') \geqslant x(m) + 1$, which contradicts the assumption. Hence $p(m) > p(m')$. Therefore $y(m) \geqslant y(m') + 1$ and $w(m) \geqslant w(m') + 1$. Similarly, we have the following.

FACT 3. *Suppose* $y(m)=y(m')$. *Then we have* $p(m)<p(m')$, $x(m)\neq x(m')$ *and* $z(m)\neq z(m')$.

FACT 4. *Suppose* $z(m)=z(m')$. *Then we have* $p(m)>p(m')$, $y(m)\neq y(m')$, $y(m)\neq y(m')$ *and* $w(m)\neq w(m')$.

FACT 5. *Suppose* $w(m)=w(m')$. *Then we have* $p(m)<p(m')$, $x(m)\neq x(m')$ *and* $z(m)\neq z(m')$.

We define

$$N_1=\max\{x(i)+z(i)-1\colon 1\leqslant i\leqslant N\}.$$

$$N_2=\max\{y(i)+w(i)-1\colon 1\leqslant i\leqslant N\}.$$

It is easy to see that N_1 is the maximum length of an increasing subsequence and N_2 is the maximum length of a decreasing subsequence in p.

Let $U=\{u_1<u_2<\cdots<u_{N_1}\}$ denote a maximum increasing subsequence and $V=\{v_1<v_2<\cdots<v_{N_2}\}$ denote a maximum decreasing subsequence. Then we have the following.

FACT 6. $v_1\leqslant u_{N_1}$, $u_1\leqslant v_{N_2}$.

Proof. Suppose $v_1>u_{N_1}$. If $p(u_{N_1})<p(v_1)$, then $u_1, u_2,..., u_{N_1}, v_1$ is an increasing subsequence, which contradicts the definition of N_1. We may assume $p(u_{N_1})>p(v_1)$. Then $u_{N_1}, v_1,..., v_{N_2}$ is a decreasing subsequence. This again is impossible. Therefore we have $v_1\leqslant u_{N_1}$. Similarly it can be shown that $u_1\leqslant v_{N_2}$.

FACT 7. *There exist j and j' such that*

$$p(u_1)\leqslant p(v_j) \qquad \textit{and} \qquad p(v_{N_2})\geqslant p(u_{j'}).$$

Proof. Suppose $p(u_1)>p(v_j)$ for any j, $1\leqslant j\leqslant N_2$. In particular we have $p(u_1)>p(v_1)$. If $v_1<u_1$, then we have an increasing subsequence v_1, $u_1,..., u_{N_1}$, which is impossible. Thus we may assume $v_1>u_1$. Then u_1, $v_1,..., u_{N_2}$ is decreasing. This contradicts the definition of N_2. Therefore $p(u_1)\leqslant p(v_j)$ for some j. Similarly it can be shown that $p(v_{N_2})\geqslant p(u_{j'})$.

FACT 8. *For any point m with $u_i<m<u_{i+1}$, we have $p(m)>p(u_{i+1})$ or $p(m)<p(u_i)$.*

Proof. This follows from the definition of U.

FACT 9. *For any point m with $v_i<m<v_{i+1}$ we have $p(m)>p(v_i)$ or $p(m)<p(v_{i+1})$.*

Proof. This follows from the definition of V.

For any number m, we define $a(m)$ to be the number of u_i's such that $u_i \leqslant m$. We define $b(m)$ to be the number of v_i's such that $v_i \leqslant m$. It is easy to see that $a(u_i) = i$ and $b(v_j) = j$.

FACT 10. *For m and m' with $m < m'$, suppose $p(m) \leqslant p(u_{a(m)})$ and $p(u_{a(m')}) < p(m')$. Then we have $z(m) + x(m') > N_1 + 1$.*

Proof. We have $x(m') \geqslant a(m') + 1$ and $z(m) \geqslant N_1 - a(m) + 1 \geqslant N_1 - a(m') + 1$. Therefore

$$z(m) + x(m') > N_1 + 1.$$

FACT 11. *For m and m' with $m < m'$, suppose $p(v_{b(m)}) < p(m)$ and $p(m') \leqslant p(v_{b(m')})$. Then we have $y(m) + w(m') > N_2 + 1$.*

Proof. Similar to the proof of Fact 10.

FACT 12. *For m and m' with $m < m'$, suppose $p(m) < p(m')$. Then $x(m) + z(m') \leqslant N_1$.*

Proof. There is an increasing subsequence of length $x(m) + z(m')$. Therefore $x(z) + z(m') \leqslant N_1$.

Similarly we have

FACT 13. *For m and m' with $m < m'$, suppose $p(m) > p(m')$. Then $w(m) + y(m') \leqslant N_2$.*

We will use x, y, z, w, a, b to define functions on $\{1,..., n\}$ which will then be used to prove the main theorem.

MAIN THEOREM. *Let ρ_n be the largest integer such that any permutation on $\{1,..., n\}$ contains a unimodal subsequence of length ρ_n. We have*

$$\rho_n = \lceil (3n - 3/4)^{1/2} - \tfrac{1}{2} \rceil.$$

This will be proved in the next section.

III. ON THE LOWER BOUND FOR ρ_n

For a fixed permutation p we want to show that $\rho(p) \geqslant (3n - 3/4)^{1/2} - \frac{1}{2}$. We first consider the case in which the maximum increasing subsequence U and the maximum decreasing subsequence contain common elements. It follows from the definition of U and V that U and V contain exactly one common number, denoted by $u_{i_*} = v_{j_*}$.

For any i with $0 \leqslant i < i_*$ we define

$$f(i) = \max_{1 \leqslant j \leqslant i+1} w(u_j) - 1. \tag{1}$$

For any j with $j \geqslant j_*$ we define

$$g(j) = \max_{k \geqslant j} z(v_k) - 1. \tag{2}$$

Now we define mappings λ and λ' from subsets of $\{1, 2, \ldots, n\}$ to the set $R = \{(i, j)\colon i + j \leqslant N - 1,\ 0 \leqslant i \leqslant N_1 - 1,\ 0 \leqslant j \leqslant N_2 - 1\}$ as follows:

Case i. $p(m) > p(u_{a(m)})$ and $p(m) > p(v_{b(m)})$, where we set $u_0 = -\infty = v_{N_2+1}$, $p(-\infty) = 0$. We define $\lambda(m) = (x(m) - 1,\ y(m) - 1)$. It is easy to see that if $m < u_{i_*}$

$$x(m) - 1 \geqslant a(m), \tag{3}$$

$$y(m) - 1 \geqslant N_2 - b(m) \geqslant N_2 - f(a(m)). \tag{4}$$

If $m > u_{i_*} = v_{j_*}$, then we have

$$x(m) - 1 \geqslant a(m) \geqslant N_1 - g(b(m)), \tag{5}$$

$$y(m) - 1 \geqslant N_2 - b(m). \tag{6}$$

These inequalities will be used to prove that λ is a one-to-one mapping and to determine a bound for the image of λ.

Case ii. $m < u_{i_*}$ and $p(u_{a(m)}) < p(m) \leqslant p(v_{b(m)})$.

We define $\lambda(m) = (x(m) - 1,\ N_2 - w(m))$. It is easily verified that

$$x(m) - 1 \geqslant a(m), \tag{7}$$

$$N_2 - b(m) \geqslant N_2 - w(m) \geqslant N_2 - f(a(m)), \tag{8}$$

$$x(m) - 1 + N_2 - w(m) \leqslant x(m) + N_2 - b(m) - 1 \leqslant N - 1. \tag{9}$$

Case iii. $m > u_{i_0}$ and $p(v_{b(m)}) < p(m) \leqslant p(u_{a(m)})$. We define $\lambda(m) = (N_1 - z(m),\ y(m) - 1)$. It is easily verified that

$$y(m) - 1 \geqslant N_2 - b(m), \tag{10}$$

$$N_1 - a(m) + 1 \leqslant z(m) \leqslant g(b(m)),$$

$$N_1 - z(m) \geqslant N_1 - g(b(m)), \tag{11}$$

$$N_1 - z(m) + y(m) - 1 \leqslant a(m) - 1 + y(m) - 1 \leqslant N - 1. \tag{12}$$

Case iv. $p(m) \leqslant p(u_{a(m)})$ and $p(m) \leqslant p(v_{b(m)})$.

If $w(m) \leqslant f(i)$ for some $i \leqslant N_1 - z(m)$, $i < i_*$, or $N_1 - z(m) \geqslant N_1 - g(j)$ for some $j \geqslant w(m)$, $j \geqslant j_*$, we define $\lambda(m) = (N_1 - z(m),\ N_2 - w(m))$; otherwise we define

$$\lambda'(m) = (z(m) - 1,\ w(m) - 1).$$

We note that

$$z(m) - 1 + w(m) - 1 \leqslant N - 1, \tag{13}$$

$$N_1 - z(m) + N_2 - w(m) \leqslant a(m) - 1 + N_2 - b(m) \leqslant N - 1. \tag{14}$$

We let M' denote the set of elements m in the domain of λ'. It is easy to see that the domain of λ is $\{1,..., n\} - M' = M$. We define

$$M_1 = \{m: \lambda(m) = (x(m) - 1,\ y(m) - 1)\},$$

$$M_2 = \{m: \lambda(m) = (x(m) - 1,\ N_2 - w(m))\},$$

$$M_3 = \{m: \lambda(m) = (N_1 - z(m),\ y(m) - 1)\},$$

$$M_4 = \{m: \lambda(m) = (N_1 - z(m),\ N_2 - w(m))\}.$$

It is easy to see that $M = M_1 \cup M_2 \cup M_3 \cup M_4$.

We will prove several properties of λ and λ' in order to establish an inequality involving N and n.

PROPERTY 1. *λ is one-to-one.*

Proof. Suppose $\lambda(m) = \lambda(m')$ and $m < m'$.

Case 1. $m, m' \in M_i$ for some i. We have a contradiction from Facts 2–5.

Case 2. $m \in M_1$, $m' \in M_2$. We have $x(m) = x(m')$ and $y(m) - 1 = N_2 - w(m')$.

From Fact 2 we have $p(m) > p(m')$. From the definition in Cases i and ii, we have $p(m) > p(v_{b(m)}) \geqslant p(v_{b(m')}) \geqslant p(m')$. By Fact 11 we have $y(m) - 1 \neq N_2 - w(m')$. This is a contradiction.

Case 3. $m \in M_2$, $m' \in M_1$. We have $x(m) = x(m')$ and $N_2 - w(m) = y(m') - 1$. From Fact 2 we have $p(m) > p(m')$. From Fact 13 we have, $N_2 - w(m) \neq y(m') - 1$, a contradiction.

Case 4. $m \in M_1$, $m' \in M_4$. We have $x(m) - 1 = N_1 - z(m')$, $y(m) - 1 = N_2 - w(m')$. Suppose $p(m) < p(m')$. From Fact 12 we have $x(m) + z(m') \leqslant N_1$, which is a contradiction. We may assume $p(m) > p(m')$. From the definition of λ in Cases i and iv, we have

$$p(m) > p(v_{b(m)}) \geqslant p(v_{b(m')}) \geqslant p(m').$$

By Fact 11, we have $y(m) + w(m') > N_2 + 1$, which is impossible.

The rest of the cases can be proved similarly by using Facts 2–5, 10–13.

PROPERTY 2. *λ' is one-to-one.*

The proof follows immediately from Fact 4. Let A_1, A_2 denote the images of λ and λ', respectively, i.e.,

$$A_1 = \{\lambda(m) : m \in M\},$$
$$A_2 = \{\lambda(m') : m' \in M'\}.$$

We define $D(i, j) = \{(i'j') : 0 \leqslant i' \leqslant N_1 - 1,\ 0 \leqslant j' \leqslant N_2 - 1,\ 0 \leqslant i' + j' \leqslant N - 1,\ i' \geqslant i, j' \geqslant j\}$.

PROPERTY 3.

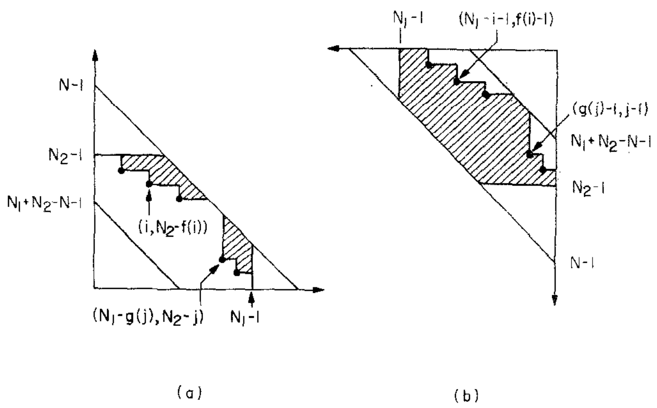
$$A_1 \subset \left(\bigcup_{i=0}^{i_*-1} D(i, N_2 - f(i)) \right) \cup \left(\bigcup_{j=j_*}^{N_2} D(N_1 - g(j), N_2 - j) \right).$$

Proof. This follows from (3)–(13) and the definition of λ in Case iv.

A point (i, j) is said to be *above* a point (i', j') if (i, j) is in $D(i', j')$. We note that any point in A_1 is above one of the points in $B = \{(i, N_2 - f(i)) : 0 \leqslant i < i_*\} \cup \{(N_1 - g(j),\ N_2 - j) : j_* \leqslant j \leqslant N_2\}$. A_1 is contained in the shaded region in Fig. 1a.

PROPERTY 4. *For $\lambda(m) \in B$ and $\lambda(m) = (i, j)$ we have $i + j \geqslant N_1 + N_2 - N$.*

Proof. For $0 \leqslant i < i_*$, $f(i) + N_1 - i \leqslant N$. Thus $i + N_2 - f(i) \geqslant N_2 + N_1 - N$. For $j_* \leqslant j \leqslant N_2$, $g(j) + j \leqslant N$. Thus $N_1 - g(j) + N_2 - j \geqslant$



(a) (b)

FIGURE 1

N_1+N_2-N. For any $(i,j)\in A$, we have $i+j\geqslant N_1+N_2-N$. From Property 3 we have $(i,j)\in A$ if $\lambda(m)=(i,j)$.

PROPERTY 5. *For $m\in M'$, we have $z(m)+w(m)\geqslant N_1+N_2-N+1$.*

Proof. From the definition of λ' in Case iv we have $p(m)<p(u_{a(m)})$ and $p(m)<p(v_{b(m)})$. We note that $N\geqslant a(m)+N_2-b(m)$. Thus $z(m)\geqslant N_1-a(m)+1$, $w(m)\geqslant b(m)$. Therefore we have

$$\begin{aligned} z(m)+w(m) &\geqslant N_1-a(m)+b(m)+1 \\ &\geqslant N_1+N_2-N+1. \end{aligned}$$

We define $D'(i,j)=\{(i',j')\colon 0\leqslant i'\leqslant N_1-1,\ 0\leqslant j'\leqslant N_2-1,\ 0\leqslant i'+j'\leqslant N-1,\ i'\leqslant i,\ j'\leqslant j\}$ and $R=\{(i,j)\colon\ i+j\leqslant N-1,\ 0\leqslant i\leqslant N_1-1,\ 0\leqslant j\leqslant N_2-1\}$.

PROPERTY 6. *Let*

$$\begin{aligned} S=&\left(\bigcup_{i=0}^{i_*-1} D'(N_1-i-1,b(i)-1)\right) \\ &\cup\left(\bigcup_{j=j_*}^{N_2} D'(g(j)-1,j-1)\right)\cup\{(i,j)\,0\leqslant i+j\leqslant N_1+N_2-N\}. \end{aligned}$$

We have $A_2\subseteq R-S$.

Proof. This follows from the definition of λ' in Case iv and Property 5.

A point (i,j) is said to be *over* a point (i',j') if (i,j) is in $D'(i',j')$. We note that any point in A_2 is not over any of the points in

$$B'=\{(N_1-i-1,f(i)-1)\colon 0\leqslant i\leqslant i_*\}\cup\{(g(j)-1,j-1)\colon j_*\leqslant j\leqslant N_2\}.$$

A_2 is contained in the shaded region in Fig. 1b. We also note that the set B' of points is just a "copy" of the set B except that B and B' are in two distinct coordinate systems. In Fig. 1, as we can see, B' is the image of B under a linear transformation. Therefore we can give an upper bound for the number of points in A_1 and A_2. From Properties 3–6, we have

$$|A_1|+|A_2|\leqslant 2\,|R|-N_1N_2.$$

Therefore

$$\begin{aligned} n &\leqslant |A_1|+|A_2| \\ &\leqslant N(1+N)-(N-N_1)\,(N-N_1+1) \\ &\quad -(N-N_2)\,(N-N_2-1)-N_1N_2 \\ &=f(N_1,N_2). \end{aligned}$$

We note that

$$\frac{\partial f}{\partial N_1}(N_1, N_2) = 1 + 2(N - N_1) - N_2$$

$$\frac{\partial f}{\partial N_2}(N_1, N_2) = 1 + 2(N - N_2) - N_1.$$

f achieves its maximum when $N_1 = N_2 = (2N+1)/3$. Thus we have

$$n \leqslant f(N_1, N_2) \leqslant \tfrac{1}{3}(N^2 + N + 1),$$

i.e.,

$$N \geqslant (3n - \tfrac{3}{4})^{1/2} - \tfrac{1}{2}. \tag{15}$$

Now, suppose the maximum increasing subsequence U and the maximum decreasing subsequence do not contain a common element. We will prove (15) by modifying the proceeding arguments as follows: Let U_{i_0} be the smallest i_0 with the property $p(u_{i_0}) \geqslant p(U_{b(u_{i0})})$ and let v_{j_0} be the smallest j_0 with the property that $p(v_{j_0}) < p(v_{a(v_{j0})})$. From Facts 6–9 and $U \cap V = \varnothing$ we have the existence of i_0, j_0 when $2 \leqslant i_0 \leqslant N_1$, $1 \leqslant j_0 \leqslant N_2$. We consider the following two possibilities.

(i) $u_{i_0} < v_{j_0}$. We set $j_* = j_0$ and $i_* = u_{a(j_*)}$. We define $f(i)$, $g(j)$ as in (1) and (2) except that we define

$$g(j_* - 1) = \max\{g(j_*), q\},$$

where

$$q = \max\{z(m): u_{i_*} < m < v_{j_*}, p(m) > p(v_{j_*})\}.$$

We define λ and λ' similarly. By considering $B^* = V \cup \{(N_1 - g(j_* - 1), N_2 - j_* + 1)\}$, the rest of the proof is an analog to that for the case in which $U \cap V \neq \varnothing$.

(ii) $u_{i_0} > v_{j_0}$. We set $i_* = i_0 - 1$, $j_* = v_{b(i_*)+1}$. We define $f(i)$, $g(j)$ as in (1) and (2) except that we define

$$f(i_*) = \max\{f(i_* - 1), q'\},$$

where

$$q' = \max\{w(m): u_{i_*} < m < v_{j_*}, p(m) > p(u_{i_*})\}.$$

We define λ and λ' similarly. By considering $B^* = B \cup \{(i_*, N_2 - f(i_*))\}$, the rest of proof is similar to that for the case in which $U \cap V \neq \varnothing$. Therefore we have proved that

$$\rho(p) \geqslant \rho_n \geqslant (3n - \tfrac{3}{4})^{1/2} - \tfrac{1}{2}. \tag{16}$$

IV. The Constructive Upper Bound

We will give explicit constructions to show that for any n there exists a permutation p on $\{1,..., n\}$ such that the longest unimodal subsequence is $\lceil (3n - \frac{3}{4})^{1/2} - \frac{1}{2} \rceil = x$. We consider the following three cases:

Case a. $x \equiv 1 \pmod 3$. We have $n = 3t^2 + 3t + 1$, where $x = 1 + 3t$. Let p_1 denote the following permutation on $\{1,..., n\}$: $p_1(j) = p_1(j-1) - 1$ except

$$p_1\left(\frac{i(2t+i+1)}{2} + 1\right) = \frac{(i+1)(2t+i+2)}{2} \qquad \text{for } i = 0, 1,..., t,$$

and

$$p_1\left(\frac{(t+1)(3t+2)}{2} + \frac{(4t-i+1)i}{2} + 1\right)$$
$$= \frac{(t+1)(3t+2)}{2} + \frac{(4t-i+2)(i+1)}{2} \qquad \text{for } i = 0,..., t-1.$$

It is straightforward to verify that the longest subsequence is $1 + 3t = x$.

Case b. $x \equiv 0 \pmod 3$. We have $n = 3t^2 + t$, where $x = 3t$. Let p_2 denote the following permutation on $\{1,..., n\}$: $p_2(j) = p_2(j-1) - 1$ except

$$p_2\left(\frac{i(2t+i-1)}{2} + 1\right) = \frac{(i+1)(2t+i)}{2} \qquad \text{for } i = 0, 1,..., t,$$

and

$$p_2\left(\frac{(t+1)3t}{2} + \frac{i(4t-i-1)}{2} + 1\right)$$
$$= \frac{(t+1)3t}{2} + \frac{(i+1)(4t-i-2)}{2} \qquad \text{for } i = 0, 1,..., t-1.$$

It can be easily checked that $\rho(p_2) = 3t = x$.

Case c. $x \equiv -1 \pmod 3$. We have $n = 3t^2 - t$, where $x = 3t - 1$. Let p_3 denote the following permutation on $\{1,..., n\}$: $p_3(j) = p(j-1) - 1$ except

$$p_3\left(\frac{i(2t+i+1)}{2} + 1\right) = \frac{(i+1)(2t+i+2)}{2} \qquad \text{for } i = 0,..., t-1,$$

and

$$p_3\left(\frac{t(3t+1)}{2}+\frac{i(4t-i-1)}{2}+1\right)$$
$$=\frac{t(3t+1)}{2}+\frac{(i+1)(4t-i-2)}{2} \qquad \text{for } i=0,\dots,t-2.$$

From Cases a, b and c we conclude that

$$\rho_n \leqslant \lceil (3n-\tfrac{3}{4})^{1/2}-\tfrac{1}{2}\rceil. \tag{17}$$

Together with (16) we complete the proof of the main theorem.

V. Concluding Remarks

The preceding result suggests a number of related problems, several of which we now mention. A subsequence $\{a_1 < a_2 < \cdots < a_t\}$ is said to be k-modal if there exist $a_1 = a_{i_0} \leqslant a_{i_1} \leqslant \cdots \leqslant a_{i_k} \leqslant a_{i_{k+1}} = a_t$ such that $\{a_{i_j} < a_{i_{j+1}} < \cdots < a_{i_{k+1}}\}$ is a monotone subsequence for all j.

We note that a monotone subsequence is 0-modal and a unimodal subsequence is 1-modal. We also note that a k-modal subsequence is $(k+1)$-modal.

1. Let $\rho(p;k)$ denote the length of the longest k-modal subsequence in p. A natural problem is to determine $\rho(n;k)$, the largest integer with the property that any permutation on $\{1,\dots,n\}$ contains a k-modal subsequence with length $\rho(n;k)$. In other words, $\rho(n;k)=\min_p \rho(p;k)$ over all permutations p on $\{1,\dots,n\}$. We know from [4] (see also [2, 3, 8]) that we have

$$\rho(n;0)=\lceil n^{1/2}\rceil.$$

In this paper we proved that

$$\rho(n;1)=\lceil (3n-\tfrac{3}{4})^{1/2}-\tfrac{1}{2}\rceil.$$

We note that $\rho(n;k)$ is an increasing function in k for fixed n. From examples similar to those in Section IV we have an upper bound for $\rho(n;k)$, namely,

$$\rho(n;k) \leqslant ((2k+1)n)^{1/2}.$$

It does not seem unreasonable to conjecture that

$$\rho(n;k)=(1+o(1))\,((2k+1)n)^{1/2}.$$

2. Let $l(n; k)$ denote the length of the longest k-modal subsequence of a random permutation, i.e.,

$$l(n; k) = \sum_{p} \rho(p; k)/n!.$$

Hammersley [5] (also see [6]) showed that

$$\lim_{n \to \infty} \frac{l(n; 0)}{n^{1/2}} = C_0$$

for some constant C_0.

Logan and Shepp [7] and Vershik and Kerov [10] confirmed the conjecture by Baer and Brock [1] by showing that $C_0 = 2$. It would be of interest to extend the above result to k-modal subsequences and the following analogous version, for any integer k, is conjectured.

$$\lim_{n \to \infty} \frac{l(n; k)}{n^{1/2}} = C_k.$$

It would also be of interest to determine the value of C_k, especially for $k = 1$.

P. Erdös posed the following problem.

Let $p = \{a_1, ..., a_n\}$ denote a sequence of real numbers satisfying $\Sigma a_i = 1$. The *sum* of a subsequence $\{a_{i_1}, ..., a_{i_t}\}$ is defined to be the values Σa_i. Let $\tau(p; k)$ denote the maximum sum of all k-modal subsequences of p. We define $\tau(n; k)$ to be the minimum value of $\tau(p, k)$ over all sequences p with n numbers. What is the value of $\tau(n; k)$?

REFERENCES

1. R. M. BAER AND P. BROCK, Natural sorting over permutation spaces, *Math. Comput.* **22** (1968), 385–410.
2. P. BLACKWELL, An alternate proof of a theorem of Erdös and Szekeres, *Amer. Math. Monthly* **78** (1971), 273.
3. V. CHVÁTAL AND J. KOMLÓS, Some combinatorial theorems on monotonicity, *Canad. Math. Bull.* **14** (1971), 151–157.
4. P. ERDÖS AND G. SZEKERES, A combinatorial problem in geometry, *Composition Math.* **2** (1935), 463–470.
5. J. M. HAMMERSLEY, A few seedlings of research, *in* "Proceedings, Sixth Berkeley Symposium Math. Stat. and Probability," Vol. 1, pp. 345–394, 1972.
6. J. F. KINGMAN, Subadditive ergodic theory, *Ann. Probability* **1** (1973), 883–889.
7. B. F. LOGAN AND L. A. SHEPP, A variational problem for Young tableaux, *Advances in Math.* **26** (1977), 206–222.

8. A. SEIDENBERG, A simple proof of a theorem of Erdös and Szekeres, *J. London Math. Soc.* **34** (1959), 352.
9. J. M. STEELE, personal communication.
0. A. M. VERSHIK AND C. V. KEROV, Asymptotics of the Plancherel measure of the symmetric group and a limiting form for Young tableaux, *Dokl. Akad. Nauk, USSR* **233** (1977), 1024–1027.